MATH 245 F21, Exam 1 Solutions

1. Carefully define the following terms: factorial, converse.

   The factorial is a *function* from $\mathbb{N}_0$ to $\mathbb{N}$, defined via: $0! = 1$, $n! = n \times (n-1)!$ (for $n \geq 1$). Given any propositions p, q, the converse of conditional proposition $p \to q$ is the *proposition* $q \to p$.

2. Carefully state the following theorems: Division Algorithm theorem, Disjunctive Syllogism semantic theorem

   The DA theorem states: Let a, b be arbitrary integers, with $b \geq 1$. Then there are unique integers q, r, satisfying $a = bq + r$ and $0 \leq r < b$. The DS theorem states: Let p, q be arbitrary propositions. If $p \vee q$ and $\neg p$ are both true, then q is true. [Or, in symbols, $p \vee q, \neg p \vdash q$.]

3. Prove or disprove: For all $a, b \in \mathbb{Z}$, if a is even and b is odd, then a^b is even.

   The statement is false. To disprove we need integers a, b, such that a is even, b is odd, and a^b is not even.

   SOLUTION 1: Take $a = 0, b = -1$. $a = 2 \cdot 0$ is even, and $b = 2 \cdot (-1) + 1$ is odd. Now $a^b = 0^{-1} = \frac{1}{0}$ is not even a number, so it's not even.

   SOLUTION 2: Take $a = 2, b = -1$. $a = 2 \cdot 1$ is even, and $b = 2 \cdot (-1) + 1$ is odd. Now $a^b = 2^{-1} = \frac{1}{2} = 0.5$ is not an integer, so it's not even.

4. Let $a, b, c \in \mathbb{Z}$, and suppose that $a|b$. Prove that $ac|bc$.

   Since $a|b$, there is some integer n with $an = b$. Multiply both sides by c, we get $anc = bc$. Rewrite as $(ac)n = bc$. Since n is (still) an integer, $ac|bc$.

5. State and prove the Conditional Interpretation Theorem.

   Thm. Let p, q be propositions. Then $p \to q \equiv q \vee \neg p$.

   Pf. The third and fifth columns of the truth table (to the right) agree; hence the two propositions are equivalent.

p	q	$p \to q$	$\neg p$	$q \vee \neg p$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

6. Simplify the proposition $\neg(p \to (q \wedge r))$ as much as possible, where only basic propositions are negated. Be sure to justify each step.

   SOLUTION 1: (1) Apply Conditional Interpretation, to get $\neg((q \wedge r) \vee \neg p)$. (2) Apply De Morgan's Law, to get $(\neg(q \wedge r)) \wedge (\neg\neg p)$. (3) Apply Double Negation, to get to get $(\neg(q \wedge r)) \wedge p$. (4) Apply De Morgan's Law, to get $((\neg q) \vee (\neg r)) \wedge p$. This is as simple as it gets, but if you want to use distributivity you can.

   SOLUTION 2: (1) Apply Negated Conditional Interpretation (Thm 2.16), to get $p \wedge \neg(q \wedge r)$. (2) Apply De Morgan's Law, to get $p \wedge ((\neg q) \vee (\neg r))$.

7. State the Modus Tollens Theorem, and prove it without truth tables. (you may use any other semantic theorems we have proved).

Thm: Let p, q be propositions. Then $p \to q, \neg q \vdash \neg p$.
Proof: All proofs begin by assuming $p \to q, \neg q$ are true. The proof ends by proving $\neg p$, which can be reached via:

METHOD 1: Apply Conditional Interpretation to get $q \vee \neg p$. Apply Disjunctive Syllogism to get $\neg p$.

METHOD 2: Apply a theorem from the book (3.13) that $p \to q$ is logically equivalent to its contrapositive, $(\neg q) \to (\neg p)$. Apply Modus Ponens to get $\neg p$.

METHOD 3: There are two cases: p can be false or true. If p is false, we are done. If instead p is true, then by modus ponens q is true. But now q is both false and true, a contradiction – so this case does not occur.

8. Let p, q, r, s be propositions. Without using truth tables, prove the following semantic theorem: $p \to (q \vee r), q \to s, r \to s \vdash p \to s$.

The proof begins by assuming $p \to (q \vee r), q \to s, r \to s$ are all true. There are many ways to proceed.

METHOD 1: Two cases: p is either false or true. If false, then addition gives us $s \vee \neg p$, and by conditional interpretation $p \to s$. If instead p is true, then modus ponens gives us $q \vee r$. We now have two subcases: if q is true, then modus ponens gives s. If instead r is true, then modus ponens again gives s. In both subcases, s is true, so addition gives us $s \vee \neg p$, and by conditional interpretation $p \to s$. In both cases, $p \to s$ is true.

METHOD 2: Two cases: s is either true or false. If true, then addition gives us $s \vee \neg p$, and by conditional interpretation $p \to s$. If instead s is false, then by modus tollens twice, we get $\neg q$ and $\neg r$. By conjunction, we get $(\neg q) \wedge (\neg r)$. By De Morgan's Law (in reverse), we get $\neg(q \vee r)$. By modus tollens one last time, we get $\neg p$. Addition gives us $s \vee \neg p$, and by conditional interpretation $p \to s$. In both cases, $p \to s$ is true.

METHOD 3: We prove $p \to s$ using a direct proof (with the added hypotheses of $p \to (q \vee r), q \to s, r \to s$). So, we assume p is true. By modus ponens, $q \vee r$ is true. We now have two cases. If q is true, by modus ponens s is true. If instead r is true, then by modus ponens again s is true. In both cases, s is true. Hence we have proved $p \to s$ using a direct proof.

9. Prove or disprove: $\forall x \in \mathbb{N},\ |4x - 9| > 1$.

The statement is false, and requires a counterexample. We take natural number $x = 2$ and find $|4x - 9| = |4 \cdot 2 - 9| = |8 - 9| = |-1| = 1$, and $1 \not> 1$. Hence $|4x - 9| \not> 1$.

10. Prove the proposition: $\exists x \in \mathbb{N},\ \forall y \in \mathbb{N},\ |0 - y| > |x - y|$.

The statement is true. First, we choose $x = 1$ (found via a side calculation[1]). Now, we let $y \in \mathbb{N}$ be arbitrary. We calculate $|0 - y| = |-y| = y$ (since $y \in \mathbb{N}$, $-y < 0$ so $|-y| = -(-y) = y$). We also calculate $|x - y| = |1 - y| = y - 1$ (since $y \in \mathbb{N}$, $y - 1 \geq 0$ and so $|1 - y| = -(1 - y) = y - 1$). Next, we calculate $(y - 1) - y = 1 \in \mathbb{N}_0$ (this proves $y \geq y - 1$, using the definitions in chapter 1), and lastly $y - 1 \neq y$ since if $y - 1 = y$ we could subtract y and get $-1 = 0$. This proves $y > y - 1$ and hence $|0 - y| > |x - y|$.

[1] This is the only choice of x that works.